The laminar and turbulent mixing of jets of compressible fluid. Part I Flow far from the orifice

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SUMMARY

For flows of jet type, the assumption of a coefficient of eddy kinematic viscosity in turbulent flow leads to the possibility of combining in one the equations for laminar and turbulent motion. An approximation to the solution of these equations is found for the flow of compressible fluid issuing from a narrow slit, far from the slit. The stream function is expanded in a power series in squares of the Mach number. Bickley's solution (1937) for the corresponding problem in incompressible flow is used to start the iterative process by which successive terms of the power series are obtained. In order to find an analytical form for the second term of the series, it has been assumed that the Prandtl number is unity, that the viscosity varies as the nth power of the absolute temperature, and that the stagnation temperature of the jet is the same as that of the surrounding gas. The solution found differs only slightly from that of Howarth (1948) and Illingworth (1949) when laminar flow is considered; only the 'change of scale' effect (arising from a distortion of the coordinates in Bickley's solution) is of importance. In turbulent flow the effect of the second term of the series is as important as the 'change of scale' effect. The effect of compressibility on the width of the mixing region is discussed for both laminar and turbulent jet flow far from the orifice.

INTRODUCTION

The problem of the mixing of streams has received a good deal of attention, both theoretically and experimentally, in recent years. The interest has been in the extension to compressible fluid flow of the problems investigated for incompressible fluid during the years before the war. For jet flows there are two main simplifications of the real problem; one is to suppose that the jet issues from a narrow slit, in which case the field at some distance from the orifice may be examined on the basis of a similarity of velocity profiles in sections normal to the axis of the jet, and the other is to treat only the so-called 'half-jet' mixing, being the case when the two streams in relative motion are both semi-infinite laterally. Each of these problems is itself divisible into two further problems, according as the flow is supposed to be laminar or turbulent.

In this paper it is first shown that the assumption of the existence of an eddy kinematic viscosity coefficient in turbulent flow, which is known to be reasonably satisfactory for jet flows, leads to the possibility of combining in one the equations for laminar and turbulent motion. Next, an approximation to the solution of these equations is found for the flow of compressible fluid issuing from a narrow slit, far from the slit. The method used is that developed by Pack (1954) for the study of axially symmetric jets of compressible fluids far from the orifice. The stream function is expanded in a power series in squares of the Mach number. The solution of Bickley (1937), for the corresponding problem in incompressible flow, is used to start the iterative process by which successive terms of the power series are obtained. In order to obtain the second term of the series in an analytical form, it has been assumed that the Prandtl number is unity, that the viscosity varies as the *n*th power of the absolute temperature, and that the stagnation temperature of the jet is the same as that of the surrounding gas. The solutions for two-dimensional flow out of a slit obtained by Howarth (1948) and Illingworth (1949) are seen as particular cases of the solutions found here. The problem of the 'half-jet' mixing will be considered in a later paper.

Equations of motion

Let u, v be the velocity components (or mean velocity components in the case of turbulent flow) parallel either to Cartesian axes (x,y) in two-dimensional flow, or to cylindrical coordinates (x,y) with x measured along the axis of symmetry in axially symmetrical flow. Let the origin of coordinates be taken at the point at which the mixing begins. Let ρ be the density of the gas in the jet and T its absolute temperature.

When the Reynolds number of the flow is large, the approximate equation of motion for jet mixing is the same as for boundary-layer flow at constant pressure:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{1}{y^\delta}\frac{\partial}{\partial y}\left(\epsilon\rho y^\delta\frac{\partial u}{\partial y}\right), \tag{1}$$

where $\delta = 0$ for two-dimensional flow and $\delta = 1$ for flow with axial symmetry. In this equation $\epsilon = \mu/\rho$, the ordinary kinematic coefficient of viscosity, when the motion is laminar, while when the flow is turbulent $\epsilon = \epsilon(x)$, a coefficient of eddy kinematic viscosity which is assumed to be independent of the y-coordinate.

The equation of continuity is

$$\frac{\partial}{\partial x}(\rho u y^\delta)+\frac{\partial}{\partial y}(\rho v y^\delta)=0. \tag{2}$$

From this equation it follows that a stream function ψ exists such that

$$\rho u y^\delta=\frac{\partial\psi}{\partial y},\qquad \rho v y^\delta=-\frac{\partial\psi}{\partial x}.$$

If the independent variables are changed to (x,z), where

$$\rho_1 z^{1+\delta}=(1+\delta)\int_0^y \rho y^\delta\,dy, \tag{3}$$

the equation of motion becomes

$$\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial x\partial z}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial z^2}+\frac{\delta}{z}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial z}=z^{\delta}\frac{\partial}{\partial z}\left\{\frac{\epsilon\rho^2y^{2\delta}}{\rho_1 z^{\delta}}\frac{\partial}{\partial z}\left(\frac{1}{z^{\delta}}\frac{\partial\psi}{\partial z}\right)\right\}. \tag{4}$$

To allow for the variation of viscosity with temperature, write

$$\frac{\mu}{\mu_1}=\mu^*=\left(\frac{T}{T_1}\right)^n=(T^*)^n. \tag{5}$$

The boundary-layer assumptions underlying the equations of motion imply that pressure is constant, to at least a first approximation, through the field, giving $\rho^*T^*=1$, where $\rho^*=\rho/\rho_1$.

Now write $\epsilon=\theta e(x)$, where (i) in a laminar jet, $\theta=\mu/\rho$ and $e(x)=1$, and (ii) in a turbulent jet, $\theta=\epsilon_0$, Reichardt's constant exchange coefficient, with $e(x)$ an experimentally determined function of x. Introduce also the new independent variable

$$\zeta=\int_0^x e(x)\,dx.$$

Then the equation of motion reduces to

$$\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial\zeta\partial z}-\frac{\partial\psi}{\partial\zeta}\frac{\partial^2\psi}{\partial z^2}+\frac{\delta}{z}\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial\zeta}=\alpha z^{\delta}\frac{\partial}{\partial z}\left\{\frac{T^{*\beta}y^{2\delta}}{z^{\delta}}\frac{\partial^2\psi}{\partial z^2}\right\}, \tag{6}$$

where $\alpha=\mu_1$ and $\beta=n-1$ for laminar flow, and $\alpha=\epsilon_0\rho_1$ and $\beta=-2$ for turbulent flow.

The energy equation is

$$\rho\left(u\frac{\partial i}{\partial x}+v\frac{\partial i}{\partial y}\right)=\frac{1}{y^{\delta}}\frac{\partial}{\partial y}\left(\frac{y^{\delta}\rho\epsilon_T}{C_p}\frac{\partial i}{\partial y}\right)+\rho\epsilon\left(\frac{\partial u}{\partial y}\right)^2, \tag{7}$$

where $\epsilon_T=\kappa/\rho$ for laminar flow, κ being the coefficient of heat conduction, and when the flow is turbulent ϵ_T is the eddy coefficient of heat transfer, and where $i=C_p T$ (the enthalpy per unit mass), C_p being the (supposed constant) specific heat at constant pressure.

When it is assumed that the Prandtl number σ, which is equal to $C_p\,\epsilon/\epsilon_T$, has the value unity, a particular integral satisfying equations (1), (2) and (7) is the Crocco relation

$$\tfrac{1}{2}u^2+i=A+Bu, \tag{8}$$

where A and B are constants, the values of which depend upon the boundary conditions in the problem considered.

Two-dimensional motion

For two-dimensional flow, $\delta=0$ in (6). In the first place it is of interest to note the simplifications of the equation of motion which are then possible.

When $T^*=1$, it follows that $z=y$, $\rho=\rho_1$ and the equation is that of incompressible flow in a boundary layer. The same equation results when $\beta=0$, this corresponding to laminar flow of a compressible fluid with viscosity proportional to temperature ($n=1$); for a two-dimensional jet, this case was treated by Howarth (1948) and Illingworth (1949).

Flow far from the orifice in a two-dimensional jet of compressible fluid

The solution for the flow far from the orifice in a two-dimensional laminar jet of incompressible fluid was given by Bickley (1937). It is

$$\psi = 6a\alpha\zeta^{1/3}\tanh a\eta, \tag{9}$$

where $\eta = z/\zeta^{2/3}$ and a is a scale factor related to the momentum flux out of the orifice. It is not difficult to show that, if F is the momentum flux per unit time out of unit length of the slit, then $a^3 = \rho_1 F/48\alpha^2$.

The same solution may clearly be used for the turbulent incompressible jet with the proper interpretation of the symbols. When $\eta = y/x$, $\zeta = x^{3/2}$, the kinematic eddy viscosity coefficient is proportional to $x^{1/2}$, and the solution is identical with that given by Görtler (1942); in this case the jet spreads linearly with x, in accordance with the experimental results of Förthmann (1934). Görtler's results were in excellent agreement with experiment except near the edges of the jet.

For the jet of compressible fluid, the boundary conditions far downstream may be expressed as follows:

$$\left.\begin{array}{llll} \text{on } y = 0, & v = 0, & \text{and } \dfrac{\partial u}{\partial y} = 0; & \\ \text{on } y = \pm\infty, & u = 0, & \dfrac{\partial u}{\partial y} = 0, & T = T_1 \text{ and } \rho = \rho_1, \end{array}\right\} \tag{10}$$

where T_1 and ρ_1 are respectively the undisturbed temperature and density in the outer medium into which the jet spreads.

When it is assumed that the stagnation temperature of the jet and the temperature of the surrounding fluid are the same, Crocco's relation (8) takes the form

$$\tfrac{1}{2}u^2 + i = i_1, \tag{11}$$

where i_1 is the stagnation enthalpy in the jet.

The solution (9) is the starting-point for the solution of the equations of compressible flow in jets. It is treated as a first approximation to the required solution. First it is seen that, from (9),

$$\rho_1 u = \frac{6a^2\alpha}{\zeta^{1/3}}\,\text{sech}^2 a\eta.$$

Insertion of this value into the Crocco relation (11) gives for the temperature

$$T^* = 1 - \frac{k^2}{\zeta^{2/3}}\,\text{sech}^4 a\eta,$$

where $k^2 = 18a^4\alpha^2/C_p\rho_1^2 T_1$. When $\epsilon \propto x^{1/2}$ and T_1 is room temperature, T^* differs from unity by less than 1% for jets of air with Mach numbers on the axis $M_A < 0{\cdot}22$. For higher Mach numbers it will not be sufficiently accurate to use the solution for the incompressible jet flow to yield quantities in the compressible jet flow. In this case the technique used by Pack (1954) for axially symmetrical laminar jets may be employed. This requires the setting-up of an iterative process similar to that of Jansen (1913) and

Rayleigh (1916) by which the approximate solution at any stage is used to obtain a better approximation

The stream function ψ is to be expanded in a series of the form

$$\psi = 6a\alpha\zeta^{1/3}\left[\tanh a\eta + \frac{k^2}{\zeta^{2/3}}F_1(a\eta) + \frac{k^4}{\zeta^{4/3}}F_2(a\eta) + \dots\right].$$

This, substituted into equation (6), leads to the following equation for F_1, on equating coefficients of k^2:

$$F_1''' + 2\tanh\xi F_1'' + 8\,\mathrm{sech}^2\xi F_1' + 4\tanh\xi\,\mathrm{sech}^2\xi F_1 = 2\beta\,\mathrm{sech}^6\xi(7\tanh^2\xi - 1),$$

where the accents denote differentiations with respect to $\xi = a\eta$. With $t = \tanh\xi$, $F_1 = (1-t^2)G$ and $(1-t^2)^2\, dG/dt = H(t)$, the above equation simplifies to

$$(1-t^2)\frac{d^2H}{dt^2} - 2t\frac{dH}{dt} + 4H = 2\beta(1-t^2)^2(7t^2-1).$$

A particular integral of this equation is the following polynomial of the sixth degree in t:

$$H = \tfrac{1}{38}\beta(17 - 72t^2 + 45t^4 - 14t^6).$$

The complementary function is a linear combination of Legendre functions:

$$H = AP_n(t) + BQ_n(t),$$

where $P_n(t) = F(-n_1, -n_2;\ 1;\ \tfrac{1}{2}(1-t))$

and $Q_n(t) = F(-n_1, -n_2;\ 1;\ \tfrac{1}{2}(1+t))$,

with n_1 and n_2 the roots of the quadratic equation $n(n+1) = 4$, the notation being the usual one for hypergeometric functions. The boundary conditions to be satisfied are:

$$\left.\begin{aligned} &\text{on } t = 0, && F_1''(0) = 0 \quad \text{and } F_1'(0) \text{ is finite;} \\ &\text{on } t = 1, && \lim_{t\to1}(1-t^2)F_1'(t) = 0, \qquad \text{i.e. } \lim_{t\to1}(1-t)F_1'(t) = 0, \\ & && \qquad \text{and } \lim_{t\to1}\{(1-t)^2F_1''(t) - (1-t)F_1'(t)\} = 0. \end{aligned}\right\}$$

When the singularities of the Legendre functions near $t = 1$ are taken into account, it follows after some analysis that $A = B = 0$. Thus, finally, in the second approximation, when F_1 is calculated from the value obtained for H

$$\psi = 6a\alpha\zeta^{1/3}\left[\tanh\xi - \frac{\beta k^2}{38\zeta^{2/3}}\left\{12\tanh\xi - \mathrm{sech}^2\xi\left(12\xi + 17\tanh\xi - \frac{14}{3}\tanh^3\xi\right)\right\}\right].$$

A check on the error involved in the use of the second approximation shows that it is likely to be less than 1% for $M_A < 0{\cdot}52$. In the neighbourhood of $M_A = 1$ the maximum error would be about 2% for laminar flow and 14% for turblent flow. These estimates are based on the assumption that F_2' is of the same order of magnitude as F_1' on the axis of the jet.

Thus the second approximation will express the effect of compressibility with fair accuracy up to sonic speed on the axis. For values of M_A greater than unity the solution still formally applies, but the possibility of the occurrence of shock-waves may render it invalid.

The values of F_1, F_1' for various values of t are given in table 1.

The correspondence between y and z (or ξ), to the first order in k^2, requires only the solution for incompressible flow. In fact,

$$ay = \zeta^{2/3}\left[\xi - \frac{k^2}{\zeta^{2/3}}\tanh\xi\left(1 - \frac{1}{3}\tanh^2\xi\right)\right].$$

Thus

$$ay \to \zeta^{2/3}\left(\xi - \frac{2}{3}\frac{k^2}{\zeta^{2/3}}\right) \quad \text{as } \xi \to \infty.$$

This relation is naturally the same as that obtained by Illingworth for the two-dimensional laminar jet, his solution corresponding to the first term alone in the expansion of ψ, for reasons already explained above.

t	$\xi = \tanh^{-1} t$	$57(-2/\beta)F_1(\xi)$	$57(-2/\beta)F_1'(\xi)$
0	0	0	−51·000
0·1	0·100	−5·012	−47·852
0·2	0·203	−9·490	−38·777
0·3	0·310	−12·919	−24·871
0·4	0·424	−14·794	−7·888
0·5	0·549	−14·643	9·863
0·6	0·693	−12·019	25·646
0·7	0·867	−6·482	36·437
0·8	1·099	2·454	39·027
0·9	1·472	15·548	29·720
1·0	∞	36·000	0

Table 1.

The difference between the velocity profile obtained from the *first term* in the expansion of ψ, when z is expressed in terms of the physical ordinate y, and the velocity profile in incompressible flow (which comes from the same term with $z = y$) will be called the 'change of scale' effect. The above expression for y in terms of ξ and ζ shows that this effect tends to sharpen the jet, i.e. to decrease the width of the mixing region for both laminar and turbulent jets; e.g. when $M_A = 1{\cdot}0$ the 'change of scale' effect diminishes the width of the mixing region by about 20%—the width of the jet being defined as $2\zeta^{2/3}/a$. The numerical value of F_1' is at most about 0·107 for a laminar jet. Since $k^2/\zeta^{2/3} \doteqdot 0{\cdot}4M_A^2$ for a perfect gas with $\gamma = C_p/C_v = 1{\cdot}4$, the effect of the term F_1' on the velocity profile of a subsonic laminar jet is negligible. This is not true for the turbulent jet because the value of F_1' on the axis is $-17/19$. Since F_1' is negative, then changes sign and decreases to zero with increasing ξ, this term tends to make the profile of the compressible turbulent jet broader than an incompressible turbulent jet with the same axial velocity. For laminar and turbulent flow far from the orifice, the net effect of compressibility is respectively to decrease and increase the width of the mixing region.

The analysis shows that the second approximation found in this paper will be useful for flows at any subsonic velocity. Higher approximations would require attention to the terms neglected in deriving the boundary-layer equations from the full Navier–Stokes equations. The neglected terms are of a higher order than those retained in deriving the second approximation under the limitation imposed on the solution for the incompressible jet, namely $\rho_1 M_A \zeta^{2/3} \gg a^2$.

Motion with axial symmetry

When $\delta = 1$ it is possible to carry out an iterative process just as above which obtains a solution for a jet with axial symmetry. This was done by Pack (1954) for a laminar jet. The equation (6) shows that the same solution, correctly interpreted, may also be used for a turbulent jet. The solution is

$$\psi = a\zeta\left\{\frac{b^2\eta^2}{(1+\frac{1}{4}b^2\eta^2)} + F_1(\eta)\frac{k^2}{\zeta^2} + F_2(\eta)\frac{k^4}{\zeta^4} + \ldots\right\},$$

where $k^2 = 2b^4/C_p T_1 \rho_1^2$, $\eta = z/\zeta a^{1/2}$, and the relation between b and the momentum flux F across a section of the jet is given by $b^2 = 3\rho_1 F/16\pi a$. The function $F_1(\eta)$ is defined by means of a new variable $t = 1/(1+\frac{1}{4}b^2\eta^2)$:

$$F_1(t) = \frac{(6+13n)}{494}(135-30t^3-8t^4) - \left(\frac{325}{76}n + \frac{519}{494}\right)t + \left(\frac{505}{228}n - \frac{917}{1482}\right)t^2 + \\ + \frac{28}{57}(1-n)t^5 - \frac{45}{247}(6+13n)(t^2-3t)\log t.$$

A table of values of F_1 and F_1' when $n = 0{\cdot}76$ is given in Pack's paper (1954).

The same general conclusions as were found for the two-dimensional laminar jet for the effects of 'change of scale' and compressibility apply equally to the laminar axially symmetric jet. It is found, for axially symmetric turbulent jets, that both of the above effects are of the same order of magnitude, but just as with the two-dimensional turbulent jet the sign of $\eta^{-1}F_1'(\eta)$ on the axis is negative; as η increases $\eta^{-1}F_1'(\eta)$ changes sign and then tends to zero as $\eta \to \infty$. This behaviour of $\eta^{-1}F_1'(\eta)$ tends to broaden the velocity profile of the jet, but the 'change of scale' is more significant and the net result is that the velocity profile of the jet is narrowed as the speed rises, in contrast with two-dimensional flow.

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References

Bickley, W. G. 1937 *Phil. Mag.* (7), **23**, 727.

Förthmann, E. 1934 *Ingen.-Arch.* **5**, 42. Translated as *Nat. Adv. Comm. Aero., Wash., Tech. Mem.* no. 789 (1936).

Görtler, H. 1942 *Z. angew. Math. Mech.* **22**, 244.

Howarth, L. 1948 *Proc. Roy. Soc.* A, **194**, 16.

Illingworth, C. R. 1949 *Proc. Roy. Soc.* A, **199**, 533.

Jansen, O. 1913 *Phys. Zeits.* **14**, 639,

Pack, D. C. 1954 *Proc. Camb. Phil. Soc.* **50**, 98.

Rayleigh, Lord 1916 *Phil. Mag.* (6), **32**, 1.